Exercises

In Exercises 1–14, to establish a big-O relationship, find witnesses $c$ and $k$ such that $|f(x)| < cg(x)$ whenever $x > k$.

1. Determine whether each of these functions is $O(x)$.
   a) $f(x) = 10$
   b) $f(x) = 3x + 7$
   c) $f(x) = x^2 + x + 1$
   d) $f(x) = 5\log x$
   e) $f(x) = [x]$  
   f) $f(x) = |x|/2$

2. Determine whether each of these functions is $O(x^2)$.
   a) $f(x) = 17x + 11$
   b) $f(x) = x^2 + 1000$
   c) $f(x) = x \log x$
   d) $f(x) = x^2/2$
   e) $f(x) = 2x^2$
   f) $f(x) = x^2 \cdot [x]$

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $x^2 + 9x + 4x + 7$ is $O(x^2)$.

4. Use the definition of "$f(x)$ is $O(g(x))$" to show that $2x + 17$ is $O(3x)$.

5. Show that $(x^2 + 1)/(x + 1)$ is $O(x)$.

6. Show that $(x^2 + 2x)/(2x + 1)$ is $O(x^2)$.

7. Find the least integer $n$ such that $f(x)$ is $O(x^n)$ for each of these functions.
   a) $f(x) = 2x^2 + x^3 \log x$
   b) $f(x) = 3x^3 + (\log x)^3$
   c) $f(x) = (x^2 + x^3 + 1)/(x^3 + 1)$
   d) $f(x) = (x^2 + 5 \log x)/(x^4 + 1)$

8. Find the least integer $n$ such that $f(x)$ is $O(x^n)$ for each of these functions.
   a) $f(x) = 2x^2 + x^3 \log x$
   b) $f(x) = 3x^3 + (\log x)^3$
   c) $f(x) = (x^2 + x^3 + 1)/(x^4 + 1)$
   d) $f(x) = (x^2 + 5 \log x)/(x^4 + 1)$

9. Show that $x^2 + 9x + 4x + 7$ is $O(x^2)$ but that $x^3$ is not $O(x^2)$.
   c) $g(x) = x^2$
   d) $g(x) = x^2 + x^3$
   e) $g(x) = 3x^3$
   f) $g(x) = x^2/2$

10. Explain what it means for a function to be $O(1)$.

11. Show that if $f(x)$ is $O(x)$, then $f(x)$ is $O(x^2)$.

12. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x) = O(g(x))$ and $g(x) = O(h(x))$. Show that $f(x)$ is $O(h(x))$.

13. Let $k$ be a positive integer. Show that $k^3 + 2k^2 + \ldots + n^k$ is $O(n^k)$.

14. Give a good big-O estimate as possible for each of these functions.
   a) $(n^2 + 5)(n^2 + 1)$
   b) $(n \log n + n^2)(n^2 + 2)$
   c) $(n^4 + 2^n)(n^3 + \log(n^2 + 1))$

20. Give a big-O estimate for each of these functions. For the function $g$ in your estimate $f(x)$ is $O(g)$, use a simple function $g$ of smallest order.
   a) $(n^3 + n^2 \log n)(\log n + 1) + (17 \log n + 19)(n^3 + 2)$
   b) $(2n^3 + n^2)(n^3 + 3^n)$
   c) $(n^4 + n^2 + 5)(n^3 + 5^n)$

21. Give a big-O estimate for each of these functions. For the function $f$ in your estimate $g(x)$ is $O(g(x))$, use a simple function $g$ of the smallest order.
   a) $n(\log(n^2 + 1) + n^2 \log n)$
   b) $(n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$
   c) $n^2 + n^2$

22. For each function in Exercise 1, determine whether that function is $\Omega(x)$ and whether it is $\Theta(x)$.

23. For each function in Exercise 2, determine whether that function is $\Omega(x^2)$ and whether it is $\Theta(x^2)$.

24. a) $f(x) = 3x + 7$
   b) Show that $2x^2 + 7$ is $\Theta(x^2)$.
   c) Show that $(x^2 + 1)/2$ is $\Theta(x^2)$.
   d) Show that $\log(x^2 + 1)$ is $\Theta(\log x)$.
   e) Show that $\log(\log x)$ is $\Theta(\log \log x)$.

25. Show that $f(x)$ is $\Theta(g(x))$ if and only if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.

26. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $O(g(x))$ if and only if $g(x)$ is $\Omega(f(x))$.

27. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $\Theta(g(x))$ if and only if there are positive constants $c_1$, $C_1$, and $C_2$ such that $C_1 |g(x)| \leq |f(x)| \leq C_2 |g(x)|$ whenever $x > k$.

28. a) $f(x) = 3x^2 + x + 1$ is $\Theta(1)$ by directly finding the constants $c_1$, $C_1$, and $C_2$ in Exercise 27.
   b) Express the relationship in part (a) using a picture showing the functions $3x^2 + x + 1$, $C_1 \cdot 3x^2$, and $C_2 \cdot 3x^2$, and the constant $k$ on the x-axis, where $C_1$, $C_2$, and $k$ are the constants you found in part (a) to show that $3x^2 + x + 1$ is $\Theta(3x^2)$.

29. Express the relationship $f(x)$ is $\Theta(g(x))$ using a picture. Show the graphs of the functions $f(x)$, $C_1 |g(x)|$, and $C_2 |g(x)|$, as well as the constant $k$ on the x-axis.

30. Explain what it means for a function to be $\Omega(1)$.

31. Explain what it means for a function to be $\Theta(1)$.

32. Give a big-O estimate of the product of the first $n$ odd positive integers.

33. Show that if $f$ and $g$ are real-valued functions such that $f(x)$ is $O(g(x))$, then for every positive integer $k$, $f^k(x)$ is $O(g^k(x))$. [Note that $f^k(x) = f(x)^k$.]

34. Show that for any integer $a$ and $b$ with $a > 1$ and $b > 1$, if $f(x)$ is $O(\log_2 x)$, then $f(x)$ is $O(\log_2 x)$. 

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35. Suppose that $f(x)$ is $O(g(x))$ where $f$ and $g$ are increasing and unbounded functions. Show that $\log \left( f(x) \right)$ is $O(\log \left| g(x) \right|)$.

36. Suppose that $f(x)$ is $O(g(x))$. Does it follow that $2^{f(x)}$ is $O(2^{g(x)})$?

37. Let $f(x)$ and $g(x)$ be functions from the set of real numbers to the set of positive real numbers. Show that if $f(x)$ and $g(x)$ are both $O(1)$, then $f(x) + g(x)$ is also $O(1)$. This is still true if $f(x)$ and $g(x)$ take negative values?

38. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$. Show that $f(x)$ is $O(h(x))$.

39. If $f(x)$ and $g(x)$ are functions from the set of positive integers to the set of positive real numbers and $f(1)$ and $g(1)$ are both $O(1)$, is $f(1) + g(1)$ also $O(1)$? Either prove that it is or give a counterexample.

40. Show that if $f(x)$ and $g(x)$ are functions from the set of positive integers to the set of real numbers and $f(1)$ is $O(g(1))$ and $g(1)$ is $O(g(2))$, then $f(1) + g(1)$ is $O(g(2))$.

41. Find functions $f$ and $g$ from the set of positive integers to the set of real numbers such that $f(x)$ is not $O(g(x))$ and $g(x)$ is not $O(f(x))$ simultaneously.

42. Express the relationship $f(x) \equiv O(g(x))$ using a picture. Show the graphs of the functions $f(x)$ and $g(x)$, as well as the constant $k$ on the real axis.

43. Show that if $f(0) = O(g(0))$, $f(x) = O(g(x))$, and $f(0) \neq 0$ and $g(x) \neq 0$ for all real numbers $x > 0$, then $f(x)/g(x)$ is $O(g(x)/g(0))$.

44. Show that if $f(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n$, and $a_0 \neq 0$, then $f(x)$ is $O(x^n)$.

45. Define the notation $f(x) \equiv \Omega(g(x))$.

46. Define the notation $f(x) \equiv \Theta(g(x))$.

47. Show that $x^2 + x^3 + x^4 \in \Theta(x^4)$.

48. Show that $x^2 + x^3 + x^4 \in \Theta(x^3)$.

49. Show that $\log x \in \Theta(x)$.

50. Show that $\log x \in \Omega(x)$.

The following problems deal with another type of asymptotic notation, called little-o notation. Because little-o notation is based on the concept of limits, a knowledge of calculus is needed for these problems. We say that $f(x) = o(g(x))$ [read $f(x)$ is little-o of $g(x)$] when

$$\lim_{x \to \infty} f(x)/g(x) = 0.$$