The 3x + 1 conjecture has an interesting history and has attracted the attention of mathematicians since the 1950s. The conjecture has been raised many times and goes by many other names, including the Collatz problem, Hasse’s algorithm, Ulam’s problem, the Syracuse problem, and Kakutani’s problem. Many mathematicians have been diverted from their work to spend time attacking this conjecture. This led to the joke that this problem was part of a conspiracy to slow down American mathematical research. See the article by Jeffrey Lagarias [La85] for a fascinating discussion of this problem and the results that have been found by mathematicians attacking it.

In Chapter 3 we will describe additional open questions about prime numbers. Students already familiar with the basic notions about primes might want to explore Section 3.4, where these open questions are discussed. We will mention other important open questions throughout the book.

Additional Proof Methods

In this chapter we introduced the basic methods used in proofs. We also described how to leverage these methods to prove a variety of results. We will use these proof methods in Chapters 2 and 3 to prove results about sets, functions, algorithms, and number theory. Among the theorems we will prove is the famous halting theorem which states that there is a problem that cannot be solved using any procedure. However, there are many important proof methods besides those we have covered. We will introduce some of these methods later in this book. In particular, in Section 4.1 we will discuss mathematical induction, which is an extremely useful method for proving statements of the form \( \forall n P(n) \), where the domain consists of all positive integers. In Section 4.3 we will introduce structural induction, which can be used to prove results about recursively defined sets. We will use the Cantor diagonalization method, which can be used to prove results about the size of infinite sets, in Section 2.4. In Chapter 5 we will introduce the notion of combinatorial proofs, which can be used to prove results by counting arguments. The reader should note that entire books have been devoted to the activities discussed in this section, including many excellent works by George Pólya ([Po61], [Po71], [Po90]).

Finally, note that we have not given a procedure that can be used for proving theorems in mathematics. It is a deep theorem of mathematical logic that there is no such procedure.

Exercises

1. Prove that \( n^2 + 1 > 2^n \) when \( n \) is a positive integer with \( 1 \leq n \leq 4 \).
2. Prove that there are no positive perfect cubes less 1000 that are the sum of the cubes of two positive integers.
3. Prove that if \( x \) and \( y \) are real numbers, then \( \max(x, y) + \min(x, y) = x + y \). [Hint: Use a proof by cases, with the two cases corresponding to \( x \geq y \) and \( x < y \), respectively.]
4. Use a proof by cases to show that \( \min(a, \min(b, c)) = \min(\min(a, b), c) \) whenever \( a, b, \) and \( c \) are real numbers.
5. Prove the triangle inequality, which states that if \( x \) and \( y \) are real numbers, then \( |x| + |y| \geq |x + y| \) (where \( |x| \) represents the absolute value of \( x \), which equals \( x \) if \( x \geq 0 \) and equals \( -x \) if \( x < 0 \)).
6. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
7. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?
8. Prove that either \( 2 \cdot 10^{500} + 15 \) or \( 2 \cdot 10^{500} + 16 \) is not a perfect square. Is your proof constructive or nonconstructive?
9. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.
10. Show that the product of two of the numbers 65\(^{1000} - 8^{2001} + 3^{177}, 79^{1212} - 9^{2399} + 2^{2001}, \) and 24\(^{4993} - 5^{1992} + 1\),
1. Prove or disprove that there is a rational number \( x \) and an irrational number \( y \) such that \( x^y \) is irrational.

2. Prove or disprove that if \( a \) and \( b \) are rational numbers, then \( a^b \) is also rational.

3. Show that each of these statements can be used to express the fact that there is a unique element \( x \) such that \( P(x) \) is true. [Hint: We can also write this statement as \( \exists x P(x) \).]
   a) \( \exists x \forall y (P(y) \leftrightarrow x = y) \)
   b) \( \exists x \forall y (P(x) \land P(y) \rightarrow x = y) \)
   c) \( \exists x (P(x) \land \forall y (P(y) \rightarrow x = y)) \)

4. Show that if \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \), then there is a unique solution of the equation \( ax + b = c \).

5. Suppose that \( a \) and \( b \) are odd integers with \( a \neq b \). Show there is a unique integer \( c \) such that \( |a - c| = |b - c| \).

6. Show that if \( r \) is an irrational number, there is a unique integer \( n \) such that the distance between \( r \) and \( n \) is less than \( \frac{1}{2} \).

7. Show that if \( n \) is an odd integer, then there is a unique integer \( k \) such that \( n \) is the sum of \( k - 2 \) and \( k + 3 \).

8. Prove that given a real number \( x \) there exist unique numbers \( n \) and \( \epsilon \) such that \( x = n + \epsilon \), \( n \) is an integer, and \( 0 < \epsilon < 1 \).

9. Prove that given a real number \( x \) there exist unique numbers \( n \) and \( \epsilon \) such that \( x = n - \epsilon \), \( n \) is an integer, and \( 0 < \epsilon < 1 \).

10. Use forward reasoning to show that if \( x \) is a nonzero real number, then \( x^2 + 1/x^2 \geq 2 \). [Hint: Start with the inequality \( (x - 1/x)^2 \geq 0 \) which holds for all nonzero real numbers \( x \).]

11. The harmonic mean of two real numbers \( x \) and \( y \) equals \( 2xy/(x + y) \). By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

12. The quadratic mean of two real numbers \( x \) and \( y \) equals \( \sqrt{(x^2 + y^2)/2} \). By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

13. Write the numbers \( 1, 2, \ldots, 2n \) on a blackboard, where \( n \) is an odd integer. Pick any two of the numbers, \( j \) and \( k \), write \( |j - k| \) on the board and erase \( j \) and \( k \). Continue this process until only one integer is written on the board. Prove that this integer must be odd.

14. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Work backward, assuming that you did end up with nine zeros.]

15. Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.

16. Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.

17. Prove that there is no positive integer \( n \) such that \( n^2 + n^3 = 100 \).

18. Prove that there are no solutions in integers \( x \) and \( y \) to the equation \( 2x^2 + 5y^2 = 14 \).

19. Prove that there are no solutions in positive integers \( x \) and \( y \) to the equation \( x^4 + y^4 = 625 \).

20. Prove that there are infinitely many solutions in positive integers \( x \), \( y \), and \( z \) to the equation \( x^2 + y^2 = z^2 \). [Hint: Let \( x = m^2 - n^2 \), \( y = 2mn \), and \( z = m^2 + n^2 \), where \( m \) and \( n \) are integers.]

21. Adapt the proof in Example 4 in Section 1.6 to prove that if \( n = abc \), where \( a \), \( b \), and \( c \) are positive integers, then \( a \leq \sqrt[3]{b}, b \leq \sqrt[3]{c}, \) or \( c \leq \sqrt[3]{n} \).

22. Prove that \( \sqrt{2} \) is irrational.

23. Prove that between every two rational numbers there is an irrational number.

24. Prove that between every rational number and every irrational number there is an irrational number.

25. Let \( S = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \), where \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_n \) are orderings of two different sequences of positive real numbers, each containing \( n \) elements.
   a) Show that \( S \) takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).
   b) Show that \( S \) takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

26. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

27. Verify the 3\( x + 1 \) conjecture for these integers.
   a) 6 b) 7 c) 17 d) 21

28. Verify the 3\( x + 1 \) conjecture for these integers.
   a) 16 b) 11 c) 35 d) 113

29. Prove or disprove that you can use dominos to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

30. Prove or disprove that you can use dominos to tile a standard checkerboard with all four corners removed.

31. Prove that you can use dominos to tile a rectangular checkerboard with an even number of squares.

32. Prove or disprove that you can use dominos to tile a 5 \( \times \) 5 checkerboard with three corners removed.
43. Use a proof by exhaustion to show that a tiling using dominoes of a 4 × 4 checkerboard with opposite corners removed does not exist. [Hint: First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

*44. Prove that when a white square and a black square are removed from an 8 × 8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [Hint: Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]

45. Show that by removing two white squares and two black squares from an 8 × 8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

*46. Find all squares, if they exist, on an 8 × 8 checkerboard so that the board obtained by removing one of these square can be tiled using straight trominoes. [Hint: First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

*47. a) Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.
   b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

*48. Prove or disprove that you can tile a 10 × 10 checkerboard using straight tetrominoes.

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**Key Terms and Results**

**TERMS**

- **proposition:** a statement that is true or false
- **propositional variable:** a variable that represents a proposition
- **truth value:** true or false
- **¬p (negation of p):** the proposition with truth value opposite to the truth value of p
- **logical operators:** operators used to combine propositions
- **compound proposition:** a proposition constructed by combining propositions using logical operators
- **truth table:** a table displaying the truth values of propositions
- **p ∨ q (disjunction of p and q):** the proposition “p or q,” which is true if and only if at least one of p and q is true
- **p ∧ q (conjunction of p and q):** the proposition “p and q” which is true if and only if both p and q are true
- **p ⊕ q (exclusive or of p and q):** the proposition “p XOR q” which is true when exactly one of p and q is true
- **p → q (p implies q):** the proposition “if p, then q,” which is false if and only if p is true and q is false
- **converse of p → q:** the conditional statement q → p
- **contrapositive of p → q:** the conditional statement ¬q → ¬p
- **inverse of p → q:** the conditional statement ¬p → ¬q
- **p ↔ q (biconditional):** the proposition “p if and only if q,” which is true if and only if p and q have the same truth value
- **bit:** either a 0 or a 1
- **Boolean variable:** a variable that has a value of 0 or 1
- **bit operation:** an operation on a bit or bits
- **bit string:** a list of bits
- **bitwise operations:** operations on bit strings that operate on each bit in one string and the corresponding bit in the other string
- **tautology:** a compound proposition that is always true
- **contradiction:** a compound proposition that is always false
- **contingency:** a compound proposition that is sometimes true and sometimes false
- **consistent compound propositions:** compound propositions for which there is an assignment of truth values to the variables that makes all these propositions true
- **logically equivalent compound propositions:** compound propositions that always have the same truth values