FIGURE 4 Counting Varieties of T-Shirts.

Solution: The tree diagram in Figure 3 displays all the ways the playoff can proceed, with the winner of each game shown. We see that there are 20 different ways for the playoff to occur.

EXAMPLE 21 Suppose that “I Love New Jersey” T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution: The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts.

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) How many ways are there to pick two representatives so that one is a mathematics major and the other is a computer science major?
   b) How many ways are there to pick one representative who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) How many ways can a student answer the questions on the test if the student answers every question?
   b) How many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco? How many of these pairs involve more than one airline?

6. There are four major autoroutes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. How many bit strings are there of length eight?

11. How many bit strings of length ten both begin and end with a 1?

12. How many bit strings are there of length six or less?

13. How many bit strings with length not exceeding $n$, where $n$ is a positive integer, consist entirely of 1s?

14. How many bit strings of length $n$, where $n$ is a positive integer, start and end with 1s?

15. How many strings are there of lowercase letters of length four or less?

16. How many strings are there of four lowercase letters that have the letter x in them?

17. How many strings of five ASCII characters contain the character @ (“at” sign) at least once? (Note: There are 128 different ASCII characters.)

18. How many positive integers between 5 and 31
   a) are divisible by 3? Which integers are these?
   b) are divisible by 4? Which integers are these?
   c) are divisible by 3 and by 4? Which integers are these?
19. How many positive integers between 50 and 100
   a) are divisible by 7? Which integers are these?
   b) are divisible by 11? Which integers are these?
   c) are divisible by both 7 and 11? Which integers are these?
20. How many positive integers less than 1000
    a) are divisible by 7?
    b) are divisible by 7 but not by 11?
    c) are divisible by both 7 and 11?
    d) are divisible by either 7 or 11?
    e) are divisible by exactly one of 7 and 11?
    f) are divisible by neither 7 nor 11?
    g) have distinct digits?
    h) have distinct digits and are even?
21. How many positive integers between 100 and 999 inclusive
    a) are divisible by 7?
    b) are odd?
    c) have the same three decimal digits?
    d) are not divisible by 4?
    e) are divisible by 3 or 4?
    f) are not divisible by either 3 or 4?
    g) are divisible by 3 but not by 4?
    h) are divisible by 3 and 4?
22. How many positive integers between 1000 and 9999 inclusive
    a) are divisible by 9?
    b) are even?
    c) have distinct digits?
    d) are not divisible by 3?
    e) are divisible by 5 or 7?
    f) are not divisible by either 5 or 7?
    g) are divisible by 5 but not by 7?
    h) are divisible by 5 and 7?
23. How many strings of three decimal digits
    a) do not contain the same digit three times?
    b) begin with an odd digit?
    c) have exactly two digits that are 4s?
24. How many strings of four decimal digits
    a) do not contain the same digit twice?
    b) end with an even digit?
    c) have exactly three digits that are 9s?
25. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?
26. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?
27. How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?
39. A palindrome is a string whose reversal is identical to the string. How many bit strings of length \( n \) are palindromes?

40. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if
   a) the bride must be in the picture?
   b) both the bride and groom must be in the picture?
   c) exactly one of the bride and the groom is in the picture?

41. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
   a) the bride must be next to the groom?
   b) the bride is not next to the groom?
   c) the bride is positioned somewhere to the left of the groom?

42. How many bit strings of length seven either begin with two 0s or end with three 1s?

43. How many bit strings of length 10 either begin with three 0s or end with two 0s?

*44. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

**45. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

46. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

47. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

48. How many different initials can someone have if a person has at least two, but no more than five, different initials. Assume that each initial is one of the 26 letters of the English language.

49. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters \(*, >, <, !, +, \) and \(=\).
   a) How many different passwords are available for this computer system?
   b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
   c) If it takes one nanosecond for a hacker to check whether each possible password is your password, how long would it take this hacker to try every possible password?

50. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

51. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form X, XX, or XXX, followed by a 10-digit telephone number of the form NXX-NXX-NXXX (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?

52. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

53. How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

54. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.

55. Use a tree diagram to determine the number of subsets of \( \{3, 7, 9, 11, 24\} \) with the property that the sum of the elements in the subset is less than 28.

56. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles.

   b) Answer the question in part (a) using counting rules.

57. a) Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.

   b) Answer the question in part (a) using counting rules.

*58. Use the product rule to show that there are \( 2^n \) different truth tables for propositions in \( n \) variables.

59. Use mathematical induction to prove the sum rule for \( m \) tasks from the sum rule for two tasks.

60. Use mathematical induction to prove the product rule for \( m \) tasks from the product rule for two tasks.

61. How many diagonals does a convex polygon with \( n \) sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

62. Data are transmitted over the Internet in datagrams, which are structured blocks of bits. Each datagram...
It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that \( R(m, n) = R(n, m) \) (see Exercise 28). We also have \( R(2, n) = n \) for every positive integer \( n \geq 2 \) (see Exercise 27). The exact values of only nine Ramsey numbers \( R(m, n) \) with \( 3 \leq m \leq n \) are known, including \( R(4, 4) = 18 \). Only bounds are known for many other Ramsey numbers, including \( R(5, 5) \), which is known to satisfy \( 43 \leq R(5, 5) \leq 49 \). The reader interested in learning more about Ramsey numbers should consult [MiRo91] or [GrRoSp90].

**Exercises**

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   a) How many socks must he take out to be sure that he has at least two socks of the same color?
   b) How many socks must he take out to be sure that he has at least two black socks?

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
   a) How many balls must she select to be sure of having at least three balls of the same color?
   b) How many balls must she select to be sure of having at least three blue balls?

5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

6. Let \( d \) be a positive integer. Show that among any group of \( d + 1 \) (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by \( d \).

7. Let \( n \) be a positive integer. Show that in any set of \( n \) consecutive integers there is exactly one divisible by \( n \).

8. Show that if \( f \) is a function from \( S \) to \( T \), where \( S \) and \( T \) are finite sets with \( |S| > |T| \), then there are elements \( s_1 \) and \( s_2 \) in \( S \) such that \( f(s_1) = f(s_2) \), or in other words, \( f \) is not one-to-one.

9. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

10. Let \( (x_i, y_i), i = 1, 2, 3, 4, 5 \), be a set of five distinct points with integer coordinates in the \( xy \) plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

11. Let \( (x_i, y_i, z_i), i = 1, 2, 3, 4, 5, 6, 7, 8, 9 \), be a set of nine distinct points with integer coordinates in \( xyz \) space. Show that the midpoint of at least one pair of these points has integer coordinates.

12. How many ordered pairs of integers \( (a, b) \) are needed to guarantee that there are two ordered pairs \( (a_1, b_1) \) and \( (a_2, b_2) \) such that \( a_1 \mod 5 = a_2 \mod 5 \) and \( b_1 \mod 5 = b_2 \mod 5 \)?

13. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
   b) Is the conclusion in part (a) true if four integers are selected rather than five?

14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.
   b) Is the conclusion in part (a) true if six integers are selected rather than seven?

15. How many numbers must be selected from the set \{1, 2, 3, 4, 5, 6\} to guarantee that at least one pair of these numbers add up to \( 7 \)?

16. How many numbers must be selected from the set \{1, 3, 5, 7, 9, 11, 13, 15\} to guarantee that at least one pair of these numbers add up to \( 16 \)?

17. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?

18. Suppose that there are nine students in a discrete mathematics class at a small college.
   a) Show that the class must have at least five male students or at least five female students.
   b) Show that the class must have at least three male students or at least seven female students.

19. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.
   a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.
b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

20. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 13, 21, 3, 17.

21. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

*23. Describe an algorithm in pseudocode producing the largest increasing or decreasing subsequence of a sequence of distinct integers.

24. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.

25. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

26. Use Exercise 25 to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.

27. Show that if \( n \) is a positive integer with \( n \geq 2 \), then the Ramsey number \( R(2, n) \) equals \( n \).

28. Show that if \( m \) and \( n \) are positive integers with \( m \geq 2 \) and \( n \geq 2 \), then the Ramsey numbers \( R(m, n) \) and \( R(n, m) \) are equal.

29. Show that there are at least six people in California (population: 36 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.

30. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars, then there are two who earned exactly the same amount of money, to the penny, last year.

31. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

32. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. [Hint: It is impossible to have a computer linked to none of the others and a computer linked to all the others.]

34. Find the least number of cables required to connect eight computers to four printers to guarantee that four computers can directly access four different printers. Justify your answer.

35. Find the least number of cables required to connect 100 computers to 20 printers to guarantee that 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.

*36. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

37. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 a.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

38. Is the statement in Exercise 37 true if 24 is replaced by

a) 2?  b) 23?  c) 25?  d) 30?

39. Show that if \( f \) is a function from \( S \) to \( T \), where \( S \) and \( T \) are finite sets and \( m = |S|/|T| \), then there are at least \( m \) elements of \( S \) mapped to the same value of \( T \). That is, show that there are distinct elements \( s_1, s_2, \ldots, s_m \) of \( S \) such that \( f(s_1) = f(s_2) = \cdots = f(s_m) \).

40. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

*41. Let \( x \) be an irrational number. Show that for some positive integer \( n \) not exceeding \( n \), the absolute value of the difference between \( jx \) and the nearest integer to \( jx \) is less than \( 1/n \).

42. Let \( n_1, n_2, \ldots, n_t \) be positive integers. Show that if \( n_1 + n_2 + \cdots + n_t = t + 1 \) objects are placed into \( t \) boxes, then for some \( i, i = 1, 2, \ldots, t \), the \( i \)th box contains at least \( n_i \) objects.

*43. A proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The notation used is the same as that used in the proof in the text.

a) Assume that \( i_k \leq n \) for \( k = 1, 2, \ldots, n^2 + 1 \). Use the generalized pigeonhole principle to show that there are \( n + 1 \) terms \( a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}} \) with \( i_{k_1} = i_{k_2} = \cdots = i_{k_{n+1}} \), where \( 1 \leq k_1 < k_2 < \cdots < k_{n+1} \).

b) Show that \( a_{j+1} > a_{j+2} \) for \( j = 1, 2, \ldots, n \). [Hint: Assume that \( a_{j+1} < a_{j+2} \), and show that this implies that \( i_{k_{j+1}} > i_{k_{j+2}} \), which is a contradiction.]

c) Use parts (a) and (b) to show that if there is no increasing subsequence of length \( n + 1 \), then there must be a decreasing subsequence of this length.