4.1 #18
Let $P(n)$ be the statement $n! < n^n$, where $n$ is greater than 1.

a) What is the statement $P(2)$?

$$P(2) = 2! < 2^2$$

b) Show that $P(2)$ is true completing the basis step of the proof.

$$P(2) = 2! < 2^2$$

$$P(2) = 2 < 4$$

TRUE

c) What is the inductive hypothesis?

$$P(n) = n! < n^n$$

d) What do you need to do in order to prove the inductive step?

Show that $P(n)$ implies $P(n + 1)$.

$$P(n) \Rightarrow P(n + 1)$$

$$n! < n^n \Rightarrow (n + 1)! < (n + 1)^{n+1}$$

e) Complete the inductive step.

$$P(n) = n! < n^n$$

Multiply the left side by $n + 1$, the right side by $(n + 1) \left(\frac{(n + 1)^n}{n^n}\right)$. This is valid because $n$ is always positive and thus $\left(\frac{(n + 1)^n}{n^n}\right)$ will always be greater than 1 by a small margin and therefore $(n + 1)\left(\frac{(n + 1)^n}{n^n}\right)$ will always be greater than $n + 1$. So we have:

$$n\times(n + 1) < n^n \times (n + 1)\left(\frac{(n + 1)^n}{n^n}\right)$$

$$(n + 1)! < (n + 1)! \times(n + 1)$$

$$(n + 1)! < (n + 1)^{n+1}$$

Some simple manipulation of the equation leaves us with $(n + 1)! < (n + 1)^{n+1}$ which is the inductive step and we have proven the induction.

f) Explain why these steps show that this inequality is true whenever $n$ is an integer greater than 1

Proving the induction proves $P(n)$ implies $P(n + 1)$. In this case, $n$ can be any integer greater than 1. Since we also proved $P(2)$ is true in our base case we also know that $P(3)$ is true since $P(n)$ implies $P(n + 1)$. And since we know $P(3)$, we know $P(4)$. And
since we know P(4) we know P(5) and this chain logic goes on as n approaches positive infinity and thus proves P(n) for all integers 2 to positive infinity.

4.2 #6
a) Determine which amounts of postage can be formed using just 3 and 10 cent stamps

3, 6, 9, 10, 12, 13, 15, 16, 18, 19, 20, 21, 22, 23…
Any number ending in 0, 3, 6, 9. Any multiple of 3. Any number greater than 17

b) Prove your answer to a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.
For the values of 3, 6, 9, 10, 12, 13, 15, and 16, they are just proven manually so they won’t be worried about here. But we will show that any postage amount 18 cents or greater can be formed using 3 and 10 cent stamps using induction.

Base: 18 cents
Formed using six 3 cent stamps.
True.

For the inductive hypothesis, we will assume that a postage amount, n, has been formed using only 10 and 3 cent stamps. We will then divide this n value up into three cases, each with an appropriate action to show that postage amount n + 1 is possible:

<table>
<thead>
<tr>
<th>Case 1: There are no 10 cent stamps.</th>
<th>Case 2: There is exactly one 10 cent stamp.</th>
<th>Case 3: There are exactly two 10 cent stamps.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action: Remove three 3 cent stamps and add one 10 cent stamp.</td>
<td>Action: Remove three 3 cent stamps and add one 10 cent stamp.</td>
<td>Action: Remove the two 10 cent stamps and add seven 3 cent stamps.</td>
</tr>
</tbody>
</table>

This covers all possible cases. We can be assured that there will be no case where there is more than two 10 cent stamps because the base case, P(18) starts in case 1, P(19) is case 2, and P(20) is case 3. The cycle continues infinitely like that and whenever there are two 10 cent stamps in the solution, the next solution removes them thus there is no case where there is more than two 10 cents stamps so long as the actions described for each solution are followed.

Note that other solutions are possible.

c) Prove your answer to a) using strong induction. How does the inductive hypothesis for this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

For strong induction, three initial cases will need to be shown:
P(19) = one 10 cent stamp and three 3 cent stamps
P(20) = two 10 cent stamps

These will act as bases cases.
For the inductive hypothesis we will assume that some postage value n has been made using only 3 and 10 cent stamps. But since this is strong induction we will also assume that every postage amount from 18 to n cents has successfully formed. Based on this, a solution to \( P(n + 1) \) is proposed:

\[ P(n + 1) = P(n - 2) + \text{one 3 cent stamp} \]

Put in English, any postage amount can be created by taking the solution to the amount three cents less than the amount and adding a 3 cent stamp to it. So \( P(20) = P(18) + \text{one 3 cent stamp}, P(21) = P(19) + \text{one 3 cent stamp}, \) etc.

Since we used strong induction and assumed that everything that came before \( P(n) \) was true, we use the above solution as a strong inductive proof.

4.3 #8
Give a recursive definition of the sequence \( \{ a_n \}, n = 1, 2, 3 \ldots \) if

Giving a recursive definition to regular function is a matter of finding the relationship each step has with the previous iteration. For example you should figure out how \( a_n \) can be defined in terms of \( a_{n-1} \). So if \( a_n = n \) then you know that each iteration adds one to the previous iteration’s total. So you can recursively define \( a_n = n \) as \( a_n = a_{n-1} + 1 \).

I don’t use any particular method to find the recursive definition; I simply look at a few examples and see if I can determine the relationship between each result. If there’s some method you’re supposed to use, I don’t know it. =)

a) \( a_n = 4n - 2 \)

Recursive definition: \( a_n = a_{n-1} + 4 \)
Base case: \( a_1 = 2 \)

b) \( a_n = 1 + (-1)^n \)

Recursive definition: \( a_n = 2 - a_{n-1} \)
Base case: \( a_1 = 0 \)

c) \( a_n = n(n + 1) \)

Recursive definition: \( a_n = a_{n-1} + 2n \)
Base case: \( a_1 = 2 \)
d) \( a_n = n^2 \)

Recursive definition: \( a_n = a_{n-1} + 2n - 1 \)

Base case: \( a_1 = 1 \)

5.1 #14
How many bit strings of length \( n \), where \( n \) is a positive number, start and end with 1s.

A bit string is, of course, a series of bits, with each having two possibilities. Thus a length 2 bit string has 4 possibilities, a length 4 bit string has 16 possibilities and so on. In general a length \( n \) bit string has \( 2^n \) possible values.

In this case we have a length \( n \) bit string where the ends have to 1s. So we can look at this as a bit string of length \( n - 2 \) with two 1s stuck at the front and back. Therefore there are \( 2^{n-2} \) bit strings possible since the 1s at the ends cannot change. The only exception to this general rule is bit strings of length 1 of which there is one possible string, “1”, that satisfies the condition

5.1 #40
In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people where the bride and groom are among these 10 people, if

Note the word row implies order so order must be accounted for among the people in the photograph. However there is no order implied in who gets to be in the picture.

a) the bride must be in the picture.

\[ 6! \times (9 \text{ choose } 5) \]

The 6! represents the number of ways six people can stand in a row. With the bride already chosen, \((9 \text{ choose } 5)\) represents the number of ways to choose 5 other people out of the remaining 9 to be in the picture.

b) both the bride and groom must be in the picture.

\[ 6! \times (8 \text{ choose } 4) \]

The 6! represents the number of ways six people can stand in a row. With the bride and groom already chosen, \((8 \text{ choose } 4)\) represents the number of ways to choose 4 other people out of the remaining 8 to be in the picture.

c) exactly one of the bride and the groom must be in the picture.

\[ 6! \times 2 \times (8 \text{ choose } 5) \]

The 6! represents the number of ways six people can stand in a row. The 2 represents either the bride or the groom (not both) being the picture. With one spot already reserved for either the bride or the groom, \((8 \text{ choose } 5)\) represents the number of ways to choose 5 other people out of the remaining 8 to be in the picture.