4. Use the definition of “f(x) is O(g(x))” to show that $2^x + 17$ is O($3^x$).

To show $2^x + 17$ is O($3^x$) then we must show that the following inequality is true. To do so, we must find a C and a k that make the inequality true.

$$2^x + 17 \leq C \cdot 3^x \text{ for } x > k$$

So we can make a couple observations. It is easy to see that the right hand side is going to grow faster than the left hand side over time. However, when is x is a small number, the 17 in the left side will dominate. So we know that we can leave C as 1 because $3^x$ will grow faster than $2^x + 17$ on its own. But we have to skip over some of the smaller values of x by setting k. The smallest value of x where $2^x + 17 \leq 3^x$ is true is 3. So we should set k to 2 top get the following equation.

$$2^x + 17 \leq 1 \cdot 3^x \text{ for } x > 2$$

So answer should basically be your C and k and small argument that make the equation true for x > your chosen k.

14. Is it true that $x^3$ is O(g(x)), if g is the given function?

Essentially you’re being asked if $x^3$ is upper bounded by the following functions.

a) $g(x) = x^2$

No. No constant coefficient could ever make $x^2$ grow faster than $x^3$.

b) $g(x) = x^3$

Yes. C = 1. k = 0. Obviously $x^2$ will always equal $x^3$.

c) $g(x) = x^2 + x^3$

Yes. C = 1. k = 0. Obviously $x^3$ is going always be less than $x^2 + x^3$ when x is 1 or greater.

d) $g(x) = x^2 + x^4$

Yes. C = 1. k = 0. Obviously $x^3$ is going always be less than $x^2 + x^4$ when x is 1 or greater.

e) $g(x) = 3^x$

Yes. C = 1. k = 0. When x = 1 they are equal. From then on, $3^x$ grows faster than $x^3$. 

f) \( g(x) = \frac{x^2}{2} \)
Yes. \( C = 2, k = 0 \). Making \( C = 2 \) makes the two sides equal for all values of \( x \).

24. a) Show that \( 3x + 7 \) is \( \Theta(x) \)
To show big theta we must prove big o and big omega. So first we do big o:

\[
3x + 7 \leq C \cdot x \quad \text{for} \quad x > k
\]

We can show this by making \( C = 4 \) (if it was 3 it would always be behind the left side and any smaller would make the right side grow slower than the left) and make \( k = 6 \) as the two sides are equal when \( x = 7 \) and the right side grows fast after that.

So \( 3x + 7 \) is \( O(x) \)

Next we must show big omega by showing the following equation.

\[
3x + 7 \geq C \cdot x \quad \text{for} \quad x > k
\]

So we can leave \( C \) as 1 and \( k \) as 0 because the right side will grow slower than the left and the left gets a head start because of the 7. So we’ve shown that \( 3x + 7 \) is \( \Omega(x) \).

Since \( 3x + 7 \) is \( O(x) \) and \( \Omega(x) \) we have proven that \( 3x + 7 \) is \( \Theta(x) \).

b) Show that \( 2x^2 + x + 7 \) is \( \Theta(x^2) \).
For the rest of them I’m just going to give the Cs and ks as the process pretty much the same for all of them.

\( 2x^2 + x + 7 \) is \( O(x^2) \)
\( C = 3, k = 3 \)
\( 2x^2 + x + 7 \) is \( \Omega(x^2) \)
\( C = 1, k = 0 \)

c) Show that \( x + \frac{1}{2} \) is \( \Theta(x) \).
\( x + \frac{1}{2} \) is \( O(x) \)
\( C = 2, k = 0 \)
\( x + \frac{1}{2} \) is \( \Omega(x) \)
\( C = 1, k = 0 \)

d) Show that \( \log(x^2 + 1) \) is \( \Theta(\log_2 x) \).
\( \log(x^2 + 1) \) is \( O(\log_2 x) \)
\( C = 1, k = 0 \) (log of base 2 grows much faster than base 10)
\( \log(x^2 + 1) \) is \( \Omega(\log_2 x) \)
\( C = 1/\log_2 10, k = 0 \) (the \( C \) converts the right side to base 10)
e) Show that $\log_{10} x$ is $\Theta(\log_2 x)$.

$\log_{10} x$ is $O(\log_2 x)$

$C = 1, k = 0$ (log of base 2 grows much faster than base 10)

$\log_{10} x$ is $\Omega(\log_2 x)$

$C = 1/\log_2 10, k = 0$ (This C converts the right side to base 10 which makes the whole thing equal)

26. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $O(g(x))$ if and only if $g(x)$ is $\Omega(f(x))$.

So since the phrase “if and only if” is used, we have two implications to prove. The first is that $f(x)$ is $O(g(x))$ implies $g(x)$ is $\Omega(f(x))$. So we start with the definition of that $f(x)$ is $O(g(x))$.

\[
f(x) \leq C_1 g(x) \text{ for } x > k \quad \text{Given}
\]

\[
g(x) \geq \frac{1}{C_1} f(x) \text{ for } x > k \quad \text{Swap sides}
\]

\[
g(x) \text{ is } \Omega(f(x)) \text{ for } x > k \quad \text{Definition proved above}
\]

Next we have to prove that $g(x)$ is $\Omega(f(x))$ implies $f(x)$ is $O(g(x))$.

\[
g(x) \text{ is } \Omega(f(x)) \text{ for } x > k \quad \text{Given}
\]

\[
g(x) \geq C_2 f(x) \text{ for } x > k \quad \text{Definition of above}
\]

\[
f(x) \leq \frac{1}{C_2} g(x) \text{ for } x > k \quad \text{Swap sides}
\]

\[
f(x) \text{ is } O(g(x)) \text{ for } x > k \quad \text{Definition proven above}
\]

So we have proven the following: $f(x)$ is $O(g(x)) \iff g(x)$ is $\Omega(f(x))$ and we have that the constants in each proof are reciprocals of each other.